An Algorithm to Calculate the Minimum Integer Weights for Arbitrary Voting Games

Aaron Strauss
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MIT Department of Political Science
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Abstract

I develop an algorithm to find the minimum integer weights for any voting game, and confirm via simulation the theoretical result of proportionality. This principle of proportionality between starting resources and power was theorized 40 years ago, but has recently come under increasing fire since Baron and Ferejohn proposed their *formateur* bargaining model. This paper does not endeavor to resolve this conflict, but does give researchers a tool to further study these competing models. I locate the minimum integer weights by mapping coalitions to a constraint problem and then using integer programming techniques to satisfy the constraints while minimizing cost. This algorithm was applied to both stochastically generated data and data from European parliaments; the results were contrasted to each other and to other power indices. The current runtime of the algorithm is exponential and thus future work will focus on increasing the efficiency of the program.

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1 Introduction

Competing interests and limited resources necessitate winners and losers. In many social science fields, a challenging problem for researchers is determining the winners (and their respective pay-offs) *a priori*, given only the starting resources. This generic puzzle maps to the field of political science in the form of weighted voting games. Using techniques from computer science and mathematics, I develop a reasonably efficient algorithm to find the minimum integer weight solution for any weighted voting game. Running this algorithm on stochastic data confirms that, for sufficiently large games, minimum integer weights are proportional to *a priori* resources. With this theoretical result confirmed, researchers can use the algorithm described in this paper as a tool to calculate predicted proportional power share when examining resource allocation relationships.

Weighted voting games are scenarios in which players control certain numbers of votes, and in which some threshold of votes (usually a majority) is needed to distribute the spoils among the "winners." The winners are those who contributed their votes to cross the threshold. Manifestations of voting games include fixed voting systems (e.g., European Union, Electoral College), elected parliaments (where the players are political parties), and shareholder blocs of corporations.

As long as one player does not have enough votes to single-handedly control the outcome, competing players must join together to form coalitions of votes. A winning coalition is a group of players whose combined vote meets or exceeds the threshold. If a winning coalition would turn into a losing coalition if it lost *any* of its players, then it is called minimal winning. Enumerating these minimal winning coalitions is a key component of many operations on voting games.

1.1 Minimum Integer Weights

The minimum integer weights for a game assign a different set of votes (or weights) to the players of that game. Thus, the minimum integer weights also define a new voting game. The winning and losing coalitions for this new game must be the same as the original. As the designation "minimum integer" suggests, the new weights must all be integer values and there must be no smaller integer game that also produces the same coalitions. If all minimal winning coalitions of a game have the same total minimum integer weight, then the game is homogeneous.

Stepping back to the larger problem of determining each player's payoff (or power) in a game, notice that two sub-problems are involved in the overall voting games. First, for a player to receive any pay-off, that player must be included in the winning coalition. Second, rarely is the distribution of pay-offs predefined; thus, the players in the governing coalition have differing levels of power, which affect the proportion of the spoils they can win for themselves. There are diverging branches of literature on these two topics. Only recently, in the works of Morelli (2001) and Snyder (2001), have scholars attempted to merge these two problems.

Minimum integer weights can be useful in solving both of the above problems. When forming a government, players must form a majority coalition that agrees on the government. An analogous setting is ministry allocation, where a majority (vote-wise) of coalition members must agree to the final government setup. The latter power relationship is more subjective, since portfolios have wide ranges of powers attached to them. On the other hand, coalition formation is binary in nature: a party is either involved or not involved in the governing coalition. Because of this difference and the availability of data, this paper uses only coalition formation information to represent empirical data.

1.2 Notation

A game is made of up a quota (or threshold) q and n players, each of whom starts with s_i seats (or weight). The notation for such a game is $[q; s_1, s_2, \ldots, s_n]$. However, since this paper deals with parliaments and other majoritarian voting games, q is always a simple majority of the total weight. Thus, a "set of weights" will by itself determine a unique game. Coalitions are notated $\{p_1, p_2, \ldots, p_i\}$, where each p represents a player, the "coalition size" is i, and the "coalition weight" is the sum of the weights for each player in the coalition.[1]

2 Prior Research

2.1 The Theory Behind Coalition Building

William Gamson (1961) was the first to examine the motivations and expectations of coalition players. He assumes that each player knows how much value the other players bring to the coalition, and uses Von Nuemann and Morgenstern's (1944) work to define the minimal winning coalition. Gamson theorizes that each player in the coalition will assert a payoff in proportion to the amount of resources that player contributes to the coalition. Hence, the resulting power of each player (if they are in the coalition) should be proportional to their beginning resources. This proportionality prediction has come to be known as "Gamson's Law."

David Baron (1989) outlined a legislative bargaining process that contradicts Gamson's proportionality claim. Baron's model uses concepts from non-cooperative

[1] Coalitions with parties that lack names will be designated as $[s_1, s_2, \ldots, s_i]$. In this case, I will specify whether the combinations of congruent coalitions should be considered.

game theory and includes an ordering of proposal-makers. The general theory was developed for determining power in policy-making through the process of proposing, amending, and approving bills; however, Baron also applied his model to coalition formation. Baron argues that the proposal-maker, or *formateur* (in this case, the party that forms the coalition around it), has a disproportionate amount of power.

Massimo Morelli (1998) disagreed with Baron's method, and described another type of non-cooperative framework for legislative bargaining. As in Baron's model, there are *formateurs* that attempt to form coalitions around themselves. But Morelli assigns higher demands to the other (non-*formateur*) parties than Baron does. The result of Morelli's framework is that coalition payoffs are proportional to the number of seats each party has. Morelli's model (unlike Baron's) does not give any party higher status in three-player games with odd total weight, which makes sense from a minimal integer weight standpoint (since each party has a weight of one, no matter what the initial resources are).

Morelli (2001) later extended his model by using the minimum integer weights as the demand ratios for the parties vying to be in the coalition. He uses a "demand bargaining set" to translate parties' demand ratios to the resultant coalition. Thus, Morelli was the first to combine coalition formation with payoffs from the resultant coalition. Given this combination, Morelli finds a proportional relationship between number of seats and "combined" power for homogeneous situations. He ignores heterogeneous games, which may partially explain the discrepancy between his results and other models.

Recently, Snyder, Ting, and Ansolabehere (2001) have attempted to settle the multiple debates in the community. First, by using a theoretical analysis of infinite games, they find two types of game equilibria: (1) an "interior" equilibrium, for which power share is proportional to vote share, and (2) a "corner" equilibrium, for which power share for smaller parties is greater than what Gamson's Law predicts. Games with many "high weight" voters (or, put another way, where the quota is low compared to the largest parties) are more likely to have a corner equilibrium.

2.2 Calculating Power for Voting Games

John Von Nuemann and Oskar Morgenstern (1944) wrote an expansive book on the subject of game theory. Included in this work (Chapter 10) is the mathematical definition of both "simple game" and its solution, which are the "minimum integer weights." This solution was the first power index to be ascribed to a voting game, although numerous others would follow. Von Nuemann and Morgenstern give an algorithm and some shortcuts to find the minimum integer weights, but the final algorithm is not generalized, and, given modern problem-solving techniques, ultimately not useful.

J. R. Isabel (1955) added a key piece of the puzzle of how to calculate minimum integer weights for homogeneous games with odd total weight. For such games, Isabel designed a polynomial time check to determine if a given set of integer weights is minimal. When each party is the intersection of two winning coalitions, and these coalitions only win by one "vote" (or, more precisely, one integer weight) then the set is minimal. Isabel discovered other interesting properties of homogeneous voting games. One property that is potentially helpful in finding minimal integer weights is that the weights for homogeneous games are unique.

L.S. Shapley and Martin Shubik (1954) started with a simple game as defined by Von Nuemann and Morgenstern, but instead of defining relative power as the minimum integer weights, they created their own power index. The Shapley-Shubik index is defined as the number of ways one player can turn a losing coalition into a winning coalition divided by the total number of possible coalitions. Interestingly, to calculate Shapley-Shubik indices one must count the same coalition multiple times (since order of formation matters). Many scholars find this approach to be counter-intuitive.

Thus, John Banzhaf (1965) used the more intuitive *combinations* (where order does not matter), as opposed to Shapley-Shubik's *permutations*, to calculate a new power index. Similar to the Shapley-Shubik formulation, a player's Banzhaf power is determined by the number of times that player can change a given coalition from winning to losing or vice versa. The difference between the Shapley-Shubik and Banzhaf indices is subtle, and quantitatively the end result is similar for large games (Leech 2002).

Yet another power index was proposed by Deegan and Packel (1979). The Deegan-Packel index deals only with minimal winning coalitions. For each minimal winning coalition, each party the coalitions receives $1/s$ "points," where s is the coalition size. A party's total power is that party's proportion of the total points. Since the set of minimal winning coalitions differs from the set of winning coalitions in size and composition, the Deegan-Packel index produces startlingly divergent values from either the Shapley-Shubik or the Banzhaf index.

2.3 Computation Difficulty of Finding the Minimum Integer Weights

Switching gears to theoretical computer science, there is a class of problems called "NP," for which there is a polynomial time check to see if a given answer is actually a solution to the problem. However, for many NP problems there is no known polynomial time algorithm that solves the problem from scratch. For example, from Isabel's proof, finding the minimum integer weights of a homogeneous voting game is in NP.

Within NP, there is a subset of the hardest NP problems, called “NP-complete”; any NP problem can be reduced to an NP-complete problem. Thus if one could solve an NP-complete problem in polynomial time, one could solve any NP problem in polynomial time. A classic NP-complete problem is the traveling salesman problem. It asks: Given towns certain distances apart, is there a complete, circular route (i.e., a path that starts/ends at the same town and hits all intermediate towns exactly once) of length less than k? Given a potential solution, it is easy to verify that this solution is a complete, circular path and that its total cost is less than k. Thus, the traveling salesman problem is in NP. (A more in-depth proof is needed to show that it is NP-complete).

But a similar problem, where one finds the *minimum* cost of a route, is not in NP. It is not (currently) possible to verify a potential solution in polynomial time, because one would have to eliminate all potential paths of lower cost from contention. Thus, this “optimization version” of the traveling salesman problem is in yet another class of problems, called NP-hard. NP-hard problems are not in NP, and NP-complete problems can be reduced to any NP-hard problem. Thus, in the unlikely event that a polynomial time solution to one NP-hard problem is discovered, all NP-hard, NP-complete, and NP problems could be solved in polynomial time.

With only that knowledge, one could conjecture that the minimum integer weight problem is NP-hard. Finding the minimum integer weights is an optimization of a problem with a polynomial time check. Even so, there is more concrete evidence that designates the minimum integer weight problem as NP-hard. Yasuko Matsui and Tomomi Matsui (1998) proved that calculating both Shapley-Shubik and Banzhaf indices were NP-hard problems. They proved that Banzhaf is NP-complete by reducing the problem to the well-known, NP-complete knapsack problem. They then did the same for the Shapley-Shubik index. This result directly leads to a proof (see Section 3.3) that coalition enumeration in general is NP-hard. Thus, the only way to find an efficient algorithm for minimum integer weights would be to somehow avoid coalition enumeration.

2.4 Analogous Research in the Field of Neuroscience

Led by Professor Jehoshua Bruck, a Caltech research group is developing efficient algorithms to model the neural networks found in brains; while this would seem wholly unrelated to legislatures, the problems at hand are startlingly similar. Neural networks are the superposition of many linear threshold functions, which are functions that take in binary inputs and output a single binary number. This function is analogous to a coalition formation function that would take in a set of parties (or alternatively, a 1 if the party would be included in the coalition and 0 otherwise), and would output a 1 if the given coalition were winning and 0 if the coalition were losing. Such a coalition formation function would define a weighted voting game, and vice versa. Therefore, linear threshold functions also have minimum integer weights.

Previously, Bohossian and Bruck (1996) demonstrated how to construct a set of integer weights that was guaranteed to be minimal. Transforming the weights to a threshold function is a simple process. Unfortunately, the inverse of this process does not allow one to take an arbitrary threshold function and subsequently determine the minimal weights. Bruck's research is currently attempting to solve that problem, which is also the main focus of this paper. While Bruck's team is using attributes of threshold function to guide the search for minimal integer solution, this paper focuses on using existing techniques in mathematics and computer science. Since the overall goal of this research is to analyze the properties of voting games and their corresponding minimum integer weights, it was necessary to develop a correct algorithm quickly without much concern for efficiency. Further improvements, some perhaps along the lines of Bruck's research, will be developed in the summer and are discussed in later sections.

2.5 Empirical Evidence in Parliaments

Given the number of voting games that occur in practice, it would be folly to make theoretical predictions without verifying the result in the political arena. Eric Browne and Mark Franklin (1973) noticed the lack of empirical data in the literature and set out to confirm Gamson's Law. Browne and Franklin operationalized a party's resources as the number of seats that party controls and the payoff as the number of cabinet positions given to the party. They then analyzed 13 parliaments from 1945 to 1969, resulting in 324 data points. Correlating payoff to seats yields a coefficient on seats of 1.07 (close to the ideal unity relation), and 86% of the payoff variance is explained purely by seats held. Browne and Franklin introduce the idea of "relative weakness" to explain why smaller parties get disproportionately higher payoffs in small governing coalitions (in terms of the number of coalition members).

Eric Browne revisited the question of the relationship between coalition payoff and seats in 1980, this time with John Frendreis. They expanded the dataset of the previous study by eight years (obs=394) and corrected for "lumpiness" (incompatible denominators in total coalition votes and number of cabinet positions) and the relative weakness effect. The new analysis explained 93% of the variance of coalition payoff.

Paul Warwick and James Druckman (2001) recently attempted to reconcile Browne and Franklin's findings of a proportional relationship with the series of theoretical works (starting with Baron's) that predict a disproportionate amount of power for the formateur. Warwick and Druckman compiled the most extensive dataset to date, with 607 observations from 1945 to 1989. Their main contribution to this series of works is that they re-operationalized coalition payoffs. Realizing that some cabinet positions are more important than others, Warwick and Druckman weighted the portfolios, thus changing the dependent variable. Accounting for lumpiness and the relative weakness effect, they found a coefficient on seats of 0.987 (almost exactly unity).

However, Warwick and Druckman debunked Baron's model incorrectly. Instead of using seat share as the independent variable, they should have used a proportional power index (such as the minimum integer weights). If such an index were used, the predictive abilities of both proportional and disproportional theoretical models could have been analyzed and contrasted. Instead, by using seat share, they must account for the relative weakness effect, thereby not actually predicting with a proportional model. For instance, the game [101; 100, 100, 1] has seat shares of approximately [0.5, 0.5, 0.005], but the party with one seat has just as much power as the others. The minimum integer weights (along with any other reasonable index) predict the power share to be [0.33, 0.33, 0.33]. The algorithm presented in this paper gives researchers such as Warwick and Druckman the opportunity to calculate and use minimum integer weights in their analyses.

3 An Algorithm to Find Minimum Integer Weights

The algorithm for finding the minimum integer weights consists of two parts: (1) defining the game in terms of constraints, and (2) finding the smallest sequence of integers that satisfies those constraints. Both of these sub-problems are currently implemented in terms of NP-hard problems; thus, while I have made some modest improvements in the efficiency of the algorithm, the runtime grows exponentially with the size of the game.

3.1 Coalition Enumeration

Enumerating the coalitions of a voting game allows the game to be characterized by a set of inequalities. These inequalities will serve as the constraints for the minimization problem in the second half of the algorithm. The fewer redundant constraints used to define a problem, the more efficient the minimization problem will be; thus, the goal of the first sub-problem will be to uniquely define a game in the fewest number of constraints possible.

A simplistic approach would be to mark all possible coalition combinations as "winning" or "losing," depending on whether the total weight for the coalition is a majority. This approach yields exactly 2^n coalitions, where n is the number of parties in the game. The winning and losing coalitions for the sample game [5; 3, 2, 2, 1] are shown in Table 1.

Game		Coalitions		
Party	Seats	Coalition	Seats	Status
a	3	∅	0	Losing
b	2	{a}	3	Losing
c	2	{b}	2	Losing
d	1	{c}	2	Losing
Majority: 5		{d}	1	Losing
		{a,b}	5	Winning
		{a,c}	5	Winning
		{a,d}	4	Losing
		{b,c}	4	Losing
		{b,d}	3	Losing
		{c,d}	3	Losing
		{a,b,c}	7	Winning
		{a,b,d}	6	Winning
		{a,c,d}	6	Winning
		{b,c,d}	5	Winning
		{a,b,c,d}	8	Winning

Table 1: All Coalitions of Sample Game

For each coalition, an inequality is formed based on whether the coalition is winning or losing. Letting the seats of the *i*th party be s_i, and the majority (or quota) be q, then:

$$\sum_{i \in S} s_i \geq q \text{ where } S \text{ is a winning coalition of parties, and}$$

$$\sum_{i \in S} s_i < q \text{ where } S \text{ is a losing coalition of parties}$$

These 2^n inequalities define the weighted voting game. Any sequence of weights that meets these inequality constraints models the same game as the original sequence of seats.

Upon closer examination, this comprehensive approach to enumeration contains redundant information. First, we need not examine super-majority coalitions. In our example, since the coalition $\{a,b\}$ is winning, then clearly all coalitions that include, as a subset, the parties a and b must also be winning. The analogous argument is true for losing coalitions. Above, we can safely remove the winning coalitions of $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, and $\{a,b,c,d\}$, along with the losing coalitions of $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, from our set of "defining" coalitions; thus, we also eliminate eight constraints. The only coalitions remaining are either minimal winning coalitions or "maximal losing coalitions" (i.e., adding any party would make it winning). These are enumerated in Table 2.

Game		Coalitions		
Party	Seats	Coalition	Seats	Status
a	3	{a,b}	5	Min. Winning
b	2	{a,c}	5	Min. Winning
c	2	{a,d}	4	Max. Losing
d	1	{b,c}	4	Max. Losing
Majority: 5		{b,d}	3	Max. Losing
		{c,d}	3	Max. Losing
		{b,c,d}	5	Min. Winning

Table 2: Minimal Winning and Maximal Losing Coalitions

A second redundancy can be eliminated through the rule of complementation. For each winning coalition, the coalition that includes all of the members *not* in the winning coalition must, by definition, be losing. (This only holds for majoritarian games.) Hence, we can eliminate all losing coalitions whose complements are winning from our defining inequalities. Note, however, that this reduction does not remove all losing coalitions. The complement of “tying coalitions”—losing coalitions whose sum is exactly half of the total sum—will not be winning either. But, in this case, we know its complement must be another tying coalition as well, so we need only one inequality for each pair of tying coalitions. This procedure changes our constraints slightly:

$\sum_{i \in S} p_i \geq q$ where S is a winning coalition of parties, and

$\sum_{i \in S} p_i = q - 1$ where S is a tying coalition of parties

Implementing both these reductions results in only four inequalities, listed in Table 3.

Game		Coalitions		
Party	Seats	Coalition	Seats	Status
a	3	{a,b}	5	Min. Winning
b	2	{a,c}	5	Min. Winning
c	2	{a,d}	4	Unique Tying
d	1	{b,c,d}	5	Min. Winning
Majority: 5				

Table 3: Minimal Winning and Unique Tying Coalitions

3.2 Party Rank Reduction

Given that set of inequality constraints, it would be possible to start searching for the smallest sequence of weights that is consistent with all the constraints. This approach would correctly find the minimum integer weights. However, more information can still be gleaned from the results of coalition construction. Helpful techniques for narrowing the potential search space include: (1) finding dummy players, (2) finding interchangeable parties, and (3) ordering the parties.

A dummy player is a party that is never included in a minimal winning coalition. These parties automatically are assigned the minimal integer weight of zero. Parties with at least one seat that are in unique tying coalitions will be included in a minimal winning coalition, so it is safe to ignore tying coalitions when searching for dummy players. (There are no dummy players in the given sample game.)

Some weighted voting games include "interchangeable parties." Interchangeable parties are those that have the exact same power when forming coalitions, even though they might have a different number of *a priori* seats. In a more technical sense, parties a and b are interchangeable when:

$$\forall S \;\; (1)\ S \text{ is minumum winning, } (2)\ a \in S, \text{ and } (3)\ b \notin S \Rightarrow$$
$$(S - \{a\} + \{b\}) \text{ is also minimal winning.}$$

In the sample game, parties b and c are clearly interchangeable since they have the same number of seats. The following theorem demonstrates how to find interchangeable parties that do not have the same number of seats.

Theorem: Parties a and b are interchangeable if and only if for each coalition size, they are included in the same number of minimal winning coalitions of that size.

Proof: The "only if" direction is trivial. By the definition of interchangeable, for all minimal winning coalitions that a is in, then b must either be in that same coalition or in an analogous coalition of the same size. The same is true for all coalitions that include b. Thus, a and b will form the same number of coalitions for each coalition size.

Instead of proving the "if" direction directly, I will prove the contrapositive: If a and b are *not* interchangeable, then there exists a coalition size for which a and b belong to different numbers of coalitions of that size. Assume, without loss of generality, that there is some coalition of size k that includes a but not b and that swapping b in for a would produce a losing coalition. Thus, b has a lower weight than a. For all coalitions of size k that include b but not a,[2] swapping a in for b would produce either (1) a minimal winning coalition of size k or (2) a superset of a minimal winning coalition of size less than k (let this size be j) that includes party a. If, when swapping all of b's k-sized coalitions, only situation 1 results, then a must be in more k-sized minimal winning coalitions than b. If situation 2 results, then repeat this procedure for size j, as our initial assumption would still hold for j-sized coalitions. Thus, there must exist a coalition size for which a is in more minimal winning coalitions than b.

Note that only minimal winning coalitions, and not tying coalitions, are relevant. If party a is involved in a tying coalition and (1) the coalition does not include b, and (2) the analogous coalition including b and excluding a does not exist, then swapping a and b would necessitate a minimal winning coalition that a is in and b is not in (or vice versa). In essence, if there is a mismatch in power in the tying coalitions, this mismatch will manifest itself in the minimal winning coalitions as well.

This characteristic allows for an easy way to find interchangeable parties and order such interchangeable sets. A set of interchangeable parties that is included in more coalitions of size k must have a larger weight than the interchangeable parties that are in fewer coalitions of that size.[3] But, must all interchangeable parties have the same minimum weight? What about minimum *integer* weight? Intuition tends toward affirmative answers, but a proof is known only for the former.

Lemma 1: Given a game with interchangeable parties a and b, which have respective weights w_a and w_b, if $w_a \neq w_b$, then there exists a smaller set of weights that defines the same game.

[2] If no such coalitions exist, the proof is done since a would be in more k-sized coalitions than b.
[3] Actually, the ordering is slightly more complex; details are provided later in this section.

Proof: Let the smaller, new set of weights be the same as the original weights but with w_a' and w_b' as the weights for parties a and b instead of w_a and w_b. Assume, without loss of generality, that $w_a > w_b$. These new weights are defined as:

$w_a' = w_a - \gamma - \varepsilon$, where ε is an appropriately small positive number.
$w_b' = w_b + \gamma - \varepsilon$
$w_i' = w_i$, for i = 1, 2 … , n s.t. $i \neq a$, $i \neq b$
$\gamma = ½(w_a - w_b)$

These new sets of weights are smaller than the original by 2 * ε. To show that the new weights still define the same game, I must show that all the original winning coalitions are still winning. There are three cases of winning coalitions. First, consider winning coalitions that do not include either parties a or b. Since $w_a' + w_b' < w_a + w_b$, the winning coalitions increase their margins of victory in the new game. Next, consider winning coalitions that include either party a or party b, but not both. Since a and b are interchangeable, these types of winning coalitions come in pairs. Of the pair, the winning coalition that includes b is clearly still winning since $w_b' > w_b$ and $w_a' < w_a$. Since, in the new game, $w_a' = w_b'$, both coalitions in the pair have the same weight; thus, both are winning. Finally, consider coalitions that include both parties a and b. Each of those coalitions must have a greater weight than its complement; denote this difference as δ. If ε is chosen such that, for every δ, 2 * ε < δ, then the coalitions will still be winning even with the new, smaller weights.

Lemma 2: Interchangeable parties have the same minimum weights.

Proof: By contradiction. Assume parties a and b are interchangeable and do not have the same minimum integer weight. Then by Lemma 1, there exists a smaller set of weights that defines the same game. Hence, the original weights are not minimum.

Theorem: Interchangeable parties have the same minimum integer weights.

Proof: A rigorous proof is currently unknown. Intuitively, because of the same principles shown above, minimal integer weights would include the same weight for interchangeable parties. Also, no counterexample can be found. Even if the theorem were not true, this characteristic would hold in the vast majority of cases. For the remainder of this paper, the theorem is assumed to be true.

A proof can almost be done using the same process as Lemma 1. In this case, I attempt to prove that there is an integer set of weights smaller than any set of *integer* weights in which $w_a \neq w_b$. To end with a smaller integer set of weights the following transformations need to be done (again assuming $w_a > w_b$):

$w_a' = w_a - 1$
$w_b' = w_b$
$w_i' = w_i$, for i = 1, 2 … , n s.t. $i \neq a$, $i \neq b$

If the new weights define the same game, then a smaller set of integer weights exists. As before, there are three types of coalitions to consider. First, coalitions that include neither parties a nor b are still winning, since $w_a' + w_b' < w_a + w_b$. Second, consider coalitions that include either parties a or b, but not both. Using similar logic as above, the coalition that includes b and not a is still winning, since $w_b' = w_b$ and $w_a' < w_a$. The second coalition in the pair, which includes a but not b, remains winning since (1) $w_a' \geq w_b'$ and (2) the first coalition in the pair is winning.

The proof fails on the third type of coalition, which has both parties a and b as members. If the coalition won by one vote with the original set of weights, then since $w_a' + w_b' = w_a + w_b - 1$, the coalition would be losing in the new scheme. The fact that the transformation given above does not work in this case by no means indicates that a smaller set of integer weights (with $w_a' = w_b'$) does not exist. However, this theorem cannot be proven at this time.

Assuming this theorem to be true, there is no need to use two variables for two interchangeable parties in our constraints. Along the same lines, an ordering can be imposed on the parties. Assign the party with the most seats a rank of one. Then, assign the party with the second most seats a rank of one if it is interchangeable with the previous party, or a rank of two if it is not. Iterating this process results in each party being assigned a rank.

<u>Characteristics of Party Ranks</u>

Let n parties (excluding dummy players) be assigned ranks 1, …, r. Let w_i be the minimal integer weight assigned to parties of rank i.

1) $r \leq n$
2) $\forall i > 1, w_i \geq w_{i-1} + 1$
3) $w_1 \geq 1$

Since there are at most as many ranks as parties, it is more efficient to use ranks to define a voting game. Thus, the defining inequalities must change slightly to account for this transition. First, some coalitions become indistinguishable when defined in terms of party ranks. In the sample game, the coalitions $\{a,b\}$ and $\{a,c\}$ are now both [1,2] because a's rank is 1 and both b's and c's ranks are 2. (Note that coalitions are no longer sets since they can contain duplicate values.) Eliminating these redundancies results in only unique rank coalitions. Also, since in the search for the minimal integer weights, the quota will change when any party weight changes, q is removed from the coalition constraints.

<u>Final Coalition Constraint Equations</u>

$\sum_{i \in S} w_i - \sum_{i \notin S} w_i \geq 1$ where S is a minimal winning coalition
$\sum_{i \in S} w_i - \sum_{i \notin S} w_i = 0$ where S is a tying coalition

In addition, the inequalities in rank characteristics two and three above can be appended to the game inequalities (see Table 4); these additional constraints narrow the search space. Thus, the total number of inequalities is equal to the sum of the number of unique-rank minimal winning coalitions, the number of unique-rank tying coalitions, and the number of ranks.

Party	Seats	Game Rank	Resulting Inequality
a	3	1	$w_1 - w_2 \geq 1$
b	2	2	$w_2 - w_3 \geq 1$
c	2	2	(none)
d	1	3	$w_3 \geq 1$
	Majority: 5		

Coalition	Seats	Coalitions Status	Resulting Inequality
[1,2]	5	Min. Winning	$w_1 - w_3 \geq 1$
[1,3]	4	Unique Tying	$w_1 + w_3 - 2w_2 = 0$
[2,2,3]	5	Min. Winning	$2w_2 + w_3 - w_1 \geq 1$

Table 4: Unique Rank Coalitions, Ranks, and Inequalities

3.3 Coalition Enumeration is NP-hard

To prove that coalition enumeration is NP-hard I reduce the known NP-hard problem of calculating Banzhaf power indices to coalition enumeration. The input to Banzhaf is the same as the input to coalition enumeration—namely, a weighted voting game. The first step in solving Banzhaf would be to enumerate the minimal winning coalitions. Then, for each party in each coalition, the numerator of that party's Banzhaf score would be incremented.[4] Thus, coalition enumeration cannot be easier to solve than Banzhaf. Coalition enumeration is NP-hard.

[4] For this reduction to be correct it must be done in polynomial time; thus, there must be a polynomial number of minimal winning coalitions. While not yet proven, this bound is probably correct since each minimal winning coalition contributes $2^{(n-|s|)}$ winning coalitions (via the power set of all parties not in the coalition) to the overall total of about $2^{(n-1)}$ winning coalitions. Work on this question will continue in the summer.

Given this result, I am unconcerned that the current coalition enumeration algorithm runs in time $O(2^n)$. The current algorithm is just a depth-first, British-museum search on a binary tree. Each node is a party, and the children are whether to add that party to the coalition or not. Thus, no party is repeated in the coalition. If adding a party results in a tying coalition, that coalition is recorded as such. If adding a party results in a winning coalition, then the coalition is recorded and that sub-tree is pruned. If the nodes are ordered so that the largest party is decided first and the smallest party last, then all recorded winning coalitions will also be minimal winning. The depth of the tree is n; thus, the running time is $O(2^n)$.

3.4 Searching the Feasible Region for the Minimal Integer Weights

With a set of constraints, the problem of finding the minimum integer weights reduces to an integer programming (IP) problem. A common technique for solving IP problems is called branch and bound; and this method is used by the overall algoirthm. Branch and bound is a minimum-cost search over the feasible region; it uses the simplex algorithm of linear programming (LP) to determine the feasibility and children of each node.

The simplex algorithm takes a set of constraints and a cost function and returns the sequence of variables that satisfies all constraints and has minimum cost. Since, in this case, the variables are the weights of each party rank, the cost function is the sum of each weight multiplied by the number of parties that have that rank. In the example game, the cost function would be: $w_1 + 2{*}w_2 + w_3$. Minimizing that function would result in the minimum weights for the game.

To use the simplex algorithm, inequalities must first be transformed into equalities by using “slack variables.” A slack variable is a variable added to the inequality that represents the difference between the sum on the left side of the inequality and the lower bound on the right side of the equation. For instance, the first coalition constraint in the sample game would be written:

$$w_1 - w_3 + s_1 = 1 \text{ where } s_1 \text{ is the slack variable}$$

Slack variables, as well as all variables in the simplex algorithm, are inherently assumed to be nonnegative. For this instantiation of simplex, there will be as many slack variables as winning coalition constraints plus rank constraints.

Even adding these slack variables is not enough, however. For technical reasons, an *artificial variable* is needed for every constraint that began as a ≥ inequality or as an equality. Unfortunately, for the minimum weight problem, most constraints meet this criterion and thus need an artificial variable.

With the addition of slack and artificial variables, there are more variables than equations. In a linear programming problem, however, the number of non-zero variables must be at most the number of constraints. Thus, variables are separated into two categories: basic and non-basic. Basic variables are allowed to be non-zero, and the set of basic variables is called the "basis."

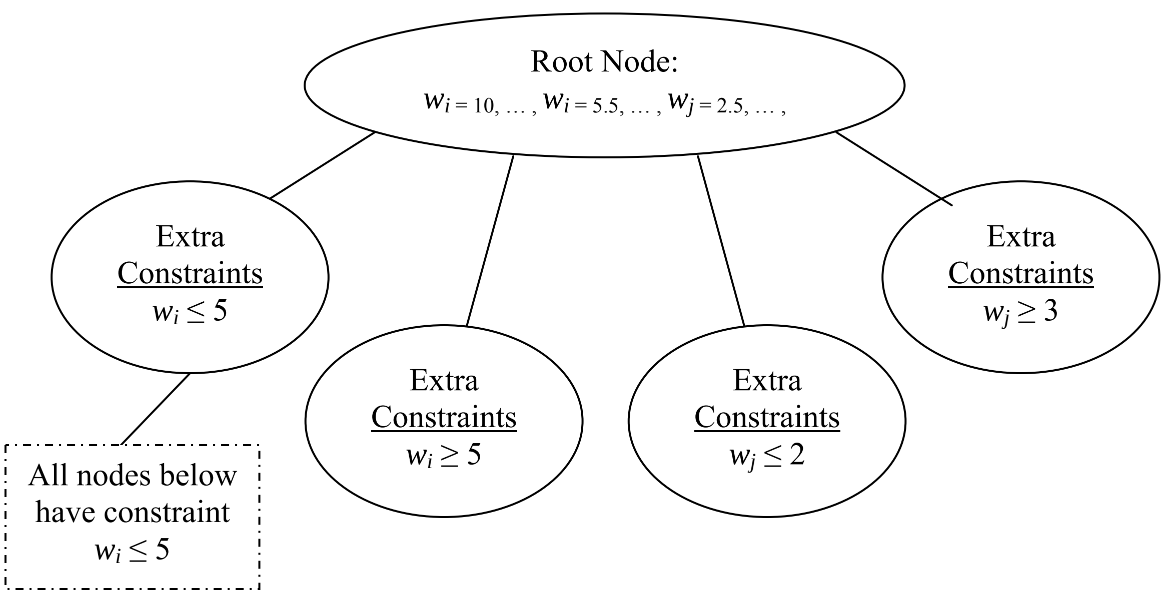
The specific type of simplex algorithm used in this paper is revised simplex (Best 1985). Revised simplex starts with all the artificial variables and the slack variables of the ≤ inequalities as the basic variables. An artificial cost function is also added, which, when optimized, ensures that all artificial variables are removed from the basis. After the artificial cost function is optimized, the original cost function is optimized. If the solution is feasible, then after those two optimizations (often called phases), all rank variables should be in the basis. The weights of the ranks are apparent from the resultant matrices.

3.5 Branch and Bound

While the set of weights calculated by the simplex algorithm is guaranteed to be minimal, the weights are not guaranteed to be integral. For homogeneous games with no tying coalitions, however, the weights actually are always integral.[5] For all other games, the IP algorithm of branch and bound is usually necessary.

Branch and bound starts by running the simplex algorithm on the initial problem. This problem is treated as the root of a search tree. If there are no non-integer weights in the solution, then the algorithm stops—those weights are the minimal integer weights. If there are non-integer weights, then for each of those weights, two additional simplex problem are produced: one in which the offending variable is constrained to be at most the floor of its current value, and another in which the variable is constrained to be at least the ceiling of its current value. Thus, if the root solution has two non-integer weights, the root node will have four children in the graph (see Figure 1).

Figure 1: Branch and Bound Algorithm



[5] Isabel's characteristics provide insight into why the simplex algorithm alone solves these games.

Each sub-problem is put in a queue, which is ordered by ascending cost. Thus, if an integer solution is found, it must be minimal, since all sub-problems of least cost have already been analyzed. The key notion is that the costs of the children of a node must be at least the cost of the parent node. Children are formed by *adding* constraints; since additional constraints can only reduce the search space, the solution to a child node cannot be more efficient than the solution to the parent. Many sub-problems will be infeasible and not have solutions; these nodes would not be included in the queue, which effectively prunes their sub-trees.

Generalized integer programming is an NP-hard problem, and thus this algorithm is too, meaning that the algorithm as a whole scales exponentially on the number of inputs (Schrijver 1987). On some inputs (some as few as ten players), branch and bound has to search over 100,000 nodes. To ensure that work is not lost, each solution is recorded in a library. The library is stored as a hashmap, with the minimal winning and unique tying rank coalitions as keys, and the minimum integer weights as values. Some non-vital data, such as the number of LP problems that had to be solved in the search process, are also recorded. Currently, the library holds over 3,000 games.

3.6 Failed Approaches

An alternative to the branch and bound method would be to cast the minimization problem as a pure search problem instead of an integer programming problem. The search space would be the different combinations of integer values for each of the ranks. Clearly this space is quite large, on the order of $(\max(p_i))^r$, since the largest minimum integer weight cannot be greater than the largest party in the initial problem, denoted by $\max(p_i)$, and since a weight needs to be found for each of the r ranks.

There are a few simple ways to lower this search space. One, the ranks are ordered, so the each weight can be prevented from outpacing (or even equaling) the weight of the next highest rank. Two, the root LP problem of the branch and bound algorithm gives a lower bound for the total weight. Thus, the first node of the search could be the integer node closest to the non-integer LP solution, and the search could be prevented from considering any set of weights with cost less than the original LP solution's cost.

These techniques, however, do not significantly shrink the number of possible solutions. Expecting that the solution would be deep in the search space (since the higher ranks can outweigh the lower ranks by an order of magnitude), a depth-first search with iterative deepening was attempted. Due to the size of the search space and the lack of intelligence built into the algorithm, this route was quickly abandoned. Those reasons also discouraged the use of stochastic processes, e.g. a search similar to WALKSAT (which is used for satisfying for Boolean logic statements).

Next, hopes were pinned on the A* search, which is regarded by scholars of artificial intelligence to be the most efficient search. To implement A*, however, a heuristic is needed to approximate the distance between the current solution and the goal. One potential candidate for a heuristic was the percent of parties that satisfied Isabel's two-coalition-intersection property. Another was the percent of constraints that the solution satisfied. The Isabel option was discarded after it was realized that while Isabel assigned this characteristic to all minimum integer weights in his paper, the characteristic is guaranteed to hold only for odd, homogeneous weights. Also, percentage heuristics in general do not completely fulfill the heuristic contract laid out by the A* search—a true A* heuristic must relate directly (and not overestimate) the remaining cost. For instance, there is no known function for translating the percent of constraints satisfied to the distance (measured in number of nodes or in cost) to the actual solution.

3.7 Future Work

A possible way to improve the efficiency of the algorithm would be to use a faster IP algorithm. Pedroso (1998) developed a niche-based genetic algorithm to solve IP problems. He claimed it ran faster than branch and bound for about 70 percent of problems. However, genetic and other randomization algorithms will not be able to prove optimality. Future work on minimum integer weights should explore IP algorithm options.

The current algorithm uses generalized techniques to solve mixed integer programming problems. The minimum integer weight problem has two specific attributes that could further constrain, and thus hasten, the search for solutions. First, all of the variables must be integral; thus, finding the weights is a pure, rather than mixed, integer programming problem. Second, the cost function is a strict addition of positive variables with integer coefficients.

While the current algorithm explores nodes with non-integral costs, the solution must have an integral cost. Thus, if the algorithm can rule out the possibility of the answer having a cost of less than or equal to k, then the algorithm can successively constrain itself to locate nodes with costs $k + 1$, $k + 2$, until it finds an integer solution. This solution would be guaranteed to be minimal.

While the bulk of the research and coding for this improved algorithm will be completed in the summer of 2003, the groundwork for such improvements has been completed. First, the linear programming problem for the root node of the current branch and bound algorithm will be calculated. As above, if this node has only integer weights, the node is the solution; otherwise, the ceiling of its cost is taken as the first *k*. Clearly, the solution cannot have cost less than *k*.[6] Thus, a constraint is added to hold the cost at *k*. (This eliminates the minimization aspect of the problem; what modifications, if any, are needed to account for this change is unknown).

For a moment, consider only homogenous games. Isabel proved that for non-tying homogeneous games, the minimum integer weight solution is unique. The same proof also implies that the solution is unique for all weights, both integral and non-integral, at the minimum cost. Thus, if the integer solution had a cost of *k*, the LP process (with the additional cost = *k* constraint) would be "forced" to return the integer solution. In the improved algorithm, for homogeneous games, only one LP problem would need to be solved at each cost step *k*. As will be shown, homogeneous games need only one iteration of the LP process in the current algorithm; hence, the above efficiency is useful only if *all* minimum integer weights are unique.

Whether minimum integer weights are unique for all voting games is currently an open question. To prove uniqueness, one would have to show strict "monotonicity" between voting weights. In other words, given a minimum integer solution, increasing one of the weights would force all other weights to either remain the same or increase as well. If other weights were capable of decreasing, then a new solution could be found with the same cost as the original, thus indicating that the solutions are not unique. Work on proving or disproving strict monotonicity will continue in the summer.

Assuming that uniqueness and monotonicity cannot be proved for (or are not characteristics of) non-homogeneous games, the integral cost step modification to the current algorithm could be implemented as follows. For each cost *k*, a new branch and bound process is started. Since the cost is constrained to add up to exactly *k*, and at each

[6] If the weight were integer and less than *k* the root simplex algorithm would have found it.

new node one variable is forced to be a different value, succeeding nodes will be feasible only if the LP algorithm is able to find an appropriate offset for the newly bounded variable. That logic, in addition to experiential evidence from the current algorithm, dictates that the branch and bound process will succeed or fail in a relatively small number of iterations.

4 Analysis and Results

4.1 Stochastic Methodology

By using this algorithm for finding the minimum integer weights for arbitrary games, I now examine different properties of games and their minimum integer weights. Characteristics examined include party weight share, the presence of dummy players, homogeneity, and algorithmic difficulty. Later sections will compare these characteristics to empirical games and other power indices.

Random games were generated based on two parameters: the number of parties and the maximum number of seats. The number of parties ranged from three to ten; the maximum seat size has five discrete values: 5, 20, 50, 100, and 250. For each party and maximum seat size, 500 random games were generated. Thus, games are laid out on two axes; I refer to games with relatively many parties in them as “large games” and games with a large maximum seat parameter as “many-seat games.” (The analogous terminology for smaller games is “small games” and “few-seat games”). Although the algorithm was unable to solve all the games in a reasonable amount of time, for each category, a majority of games was solved (see Table 5). Because the data are thus biased toward games that are easier to solve, the analysis focuses on trends in the data rather than exact values.

Number of Players	Maximum Number of Seats 5	20	50	100	250	Total
3	100%	100%	100%	100%	100%	100%
4	100%	100%	100%	100%	100%	100%
5	100%	100%	100%	100%	100%	100%
6	100%	100%	100%	100%	100%	100%
7	100%	100%	100%	100%	100%	100%
8	100%	96%	87%	90%	96%	94%
9	100%	74%	59%	69%	81%	77%
10	100%	81%	57%	56%	73%	73%
Total	100%	94%	88%	89%	94%	93%

Table 5: Percent of Games Solved

As expected, the more players in a game, the harder it is to solve. A more interesting result, though, is that after a certain level of maximum seats, games get easier to solve (see Table 6). A few potential reasons for this phenomenon were quickly ruled out. First, the number of dictator games (discussed in the next section) does not increase with a higher maximum seat count. Second, the percent of homogeneous games does not rise in many-seat games. The main reason for this downswing is that as the maximum number of seats grows very large, the smaller the probability of the game having a tying coalition. Having a tying coalition produces more minimal winning coalitions, thus increasing the time per branch and bound node. Also, as will be explored later, games with tying coalitions tend to pass through more nodes in the search.

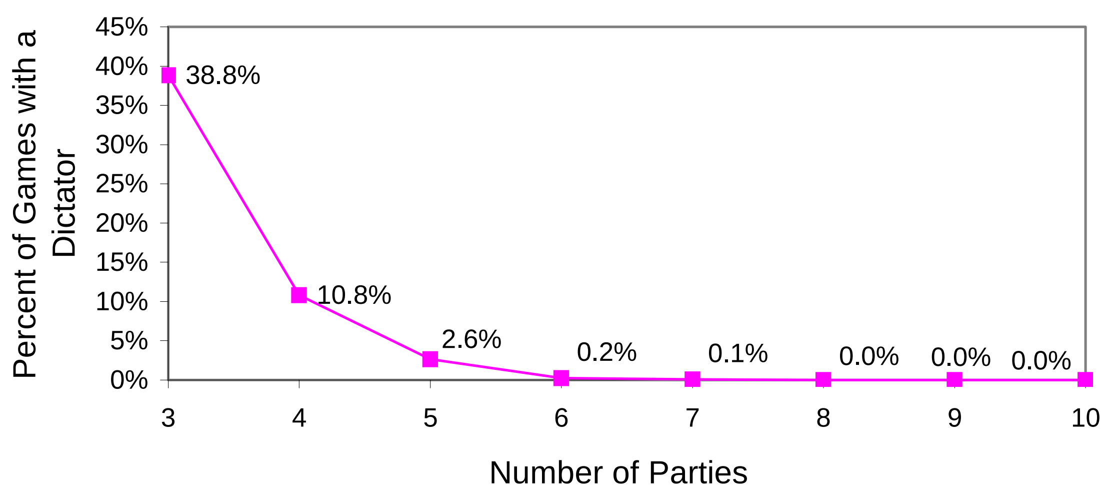
Number of Players	Maximum Number of Seats 5	20	50	100	250	Total
3	20%	9%	3%	2%	1%	7%
4	44%	12%	5%	2%	1%	13%
5	43%	19%	8%	4%	2%	15%
6	47%	31%	16%	8%	3%	21%
7	52%	43%	26%	14%	5%	28%
8	48%	46%	37%	24%	10%	33%
9	53%	49%	47%	34%	20%	41%
10	50%	48%	44%	45%	27%	43%
Total	45%	32%	23%	17%	9%	25%

Table 6: Percent of Games With Tying Coalitions

4.2 Dictator Games

When one party's weight alone is at least a majority of the total weight, that player is considered the "dictator" and the game is a "dictator game." A probabilistic analysis indicates that for three players with randomly chosen seats (no maximum), the probability of the resulting game including a dictator is 50%.[7] Indeed, almost half of the large-seat, three-player stochastic games include dictators. The data also confirm the probabilistic fact that as the number of players increases, the rarer dictator games become (see Figure 2). Since game theorists are interested only in games for which multi-player coalitions are formed, dictator games are excluded from further analyses. This proscription has a significant effect only on small-player games.

Figure 2: Incidence of Dictator Games



4.3 Weight Share vs. Seat Share Proportionality

A major impetus for this paper was to discover whether minimum integer weights, and thus potentially power, follow Gamson's Law of proportionality. Empirical evidence indicated that for games with few parties, larger parties would have

[7] To calculate this, find the probability of $\Pr(Z - W > 0) + \Pr(Z - W' > 0)$ where $W = X + Y$, $W' = |X - Y|$, and X,Y, and Z are random variables uniformly distributed over [0,1].

disproportionately less power, while for games with many parties, power was proportional. Given an algorithm to solve arbitrarily large games, whether minimum integer weights concur with this theoretical result can now be examined.

Number Of Parties	Maximum Number of Seats 5	20	50	100	250	Total
3	Coeff. = 0.339 Inter. = 0.220 R-sq. = 0.339	Coeff. = 0.160 Inter. = 0.280 R-sq. = 0.161	Coeff. = 0.072 Inter. = 0.309 R-sq. = 0.072	Coeff. = 0.048 Inter. = 0.316 R-sq. = 0.048	Coeff. = 0.022 Inter. = 0.326 R-sq. = 0.022	Coeff. = 0.142 Inter. = 0.286 R-sq. = 0.142
4	Coeff. = 1.020 Inter. = -0.005 R-sq. = 0.723	Coeff. = 0.860 Inter. = 0.035 R-sq. = 0.612	Coeff. = 0.816 Inter. = 0.046 R-sq. = 0.604	Coeff. = 0.811 Inter. = 0.047 R-sq. = 0.592	Coeff. = 0.824 Inter. = 0.044 R-sq. = 0.605	Coeff. = 0.857 Inter. = 0.036 R-sq. = 0.621
5	Coeff. = 0.952 Inter. = 0.010 R-sq. = 0.830	Coeff. = 0.916 Inter. = 0.017 R-sq. = 0.756	Coeff. = 0.882 Inter. = 0.024 R-sq. = 0. 750	Coeff. = 0.895 Inter. = 0.020 R-sq. = 0.749	Coeff. = 0.900 Inter. = 0.020 R-sq. = 0.730	Coeff. = 0.904 Inter. = 0.019 R-sq. = 0.760
6	Coeff. = 0.994 Inter. = 0.001 R-sq. = 0.926	Coeff. = 0.922 Inter. = 0.013 R-sq. = 0.870	Coeff. = 0.945 Inter. = 0.009 R-sq. = 0.850	Coeff. = 0.923 Inter. = 0.013 R-sq. = 0.840	Coeff. = 0.935 Inter. = 0.011 R-sq. = 0.842	Coeff. = 0.939 Inter. = 0.010 R-sq. = 0.858
7	Coeff. = 0.988 Inter. = 0.002 R-sq. = 0.972	Coeff. = 0.951 Inter. = 0.007 R-sq. = 0.936	Coeff. = 0.936 Inter. = 0.009 R-sq. = 0.926	Coeff. = 0.930 Inter. = 0.010 R-sq. = 0.920	Coeff. = 0.922 Inter. = 0.011 R-sq. = 0.913	Coeff. = 0.940 Inter. = .009 R-sq. = 0.929
8	Coeff. = 0.999 Inter. = 0.000 R-sq. = 0.990	Coeff. = 0.964 Inter. = 0.005 R-sq. = 0.973	Coeff. = 0.949 Inter. = 0.006 R-sq. = 0.962	Coeff. = 0.939 Inter. = 0.008 R-sq. = 0.957	Coeff. = 0.944 Inter. = 0.007 R-sq. = 0.957	Coeff. = 0.954 Inter. = 0.006 R-sq. = 0.965
9	Coeff. = 0.998 Inter. = 0.000 R-sq. = 0.996	Coeff. = 0.982 Inter. = 0.002 R-sq. = 0.988	Coeff. = 0.966 Inter. = 0.004 R-sq. = 0.982	Coeff. = 0.968 Inter. = 0.004 R-sq. = 0.981	Coeff. = 0.958 Inter. = 0.005 R-sq. = 0.979	Coeff. = 0.972 Inter. = 0.003 R-sq. = 0.984
10	Coeff. = 1.000 Inter. = 0.000 R-sq. = 0.999	Coeff. = 0.990 Inter. = 0.001 R-sq. = 0.996	Coeff. = 0.981 Inter. = 0.002 R-sq. = 0.992	Coeff. = 0.974 Inter. = 0.003 R-sq. = 0.991	Coeff. = 0.986 Inter. = 0.001 R-sq. = 0.992	Coeff. = 0.987 Inter. = 0.001 R-sq. = 0.994
Total	Coeff. = 0.953 Inter. = 0.007 R-sq. = 0.901	Coeff. = 0.914 Inter. = 0.013 R-sq. = 0.853	Coeff. = 0.899 Inter. = 0.016 R-sq. = 0.839	Coeff. = 0.902 Inter. = 0.015 R-sq. = 0.840	Coeff. = 0.904 Inter. = 0.015 R-sq. = 0.843	Coeff. = 0.913 Inter. = 0.013 R-sq. = 0.854

All coefficient and intercept numbers > .001 are significant at the 99% confidence level
Regression Equation: Seat share = *Coeff.* * Weight share + *Inter.*

Table 7: Regression Statistics for Weight Share vs. Seat Share

The data in Table 7 show the regression of weight share on seat share. Seat share is the proportion of a party's seats to the total number of seats in the game; weight share is the analogous number for the minimum integer weights. The number of data points for each entry is the number of games solved (which can be determined from Table 5, minus dictator games) multiplied by the number of parties for that entry. The minimum number of such parties in solved games is 2,637 for 9 players with a maximum of 50 seats. (This minimum does not include games of size 3, 4, or 5 players, which have fewer data points due to their lack of parties and the relative abundance of dictator games. With those games included, the minimum number of observations for a party*seat classification is 792.)

The first trend to note is that for few-seat games with a large number of parties, weight share seems to have an almost perfect correlation with seat share. This occurs because for a large proportion of such games, the original seat assignments *are* the minimum integer weights. Thus, for each party in these games, the seat share and weight share are a result of the same calculation, and thus have the same value.

While, the relationship approaches proportionality for large games, there is still a bias toward small parties (see Figure 3). For all but two categories, the coefficient on seat share is less than unity, meaning that a one percent increase in seat share will result in less than a one percent increase in weight share. Also, as Figure 4 highlights, the intercept is positive, which supports the conclusion that smaller parties have disproportionately larger weights.

Figure 3: Weight Share Regression Coefficients

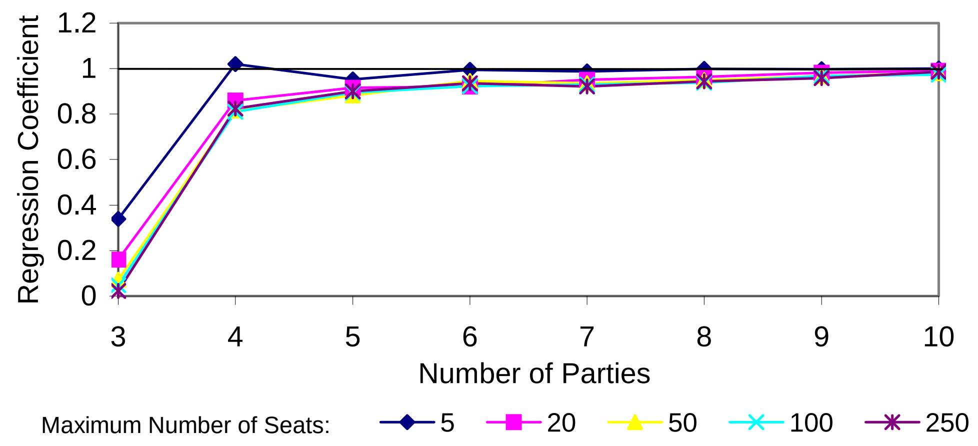
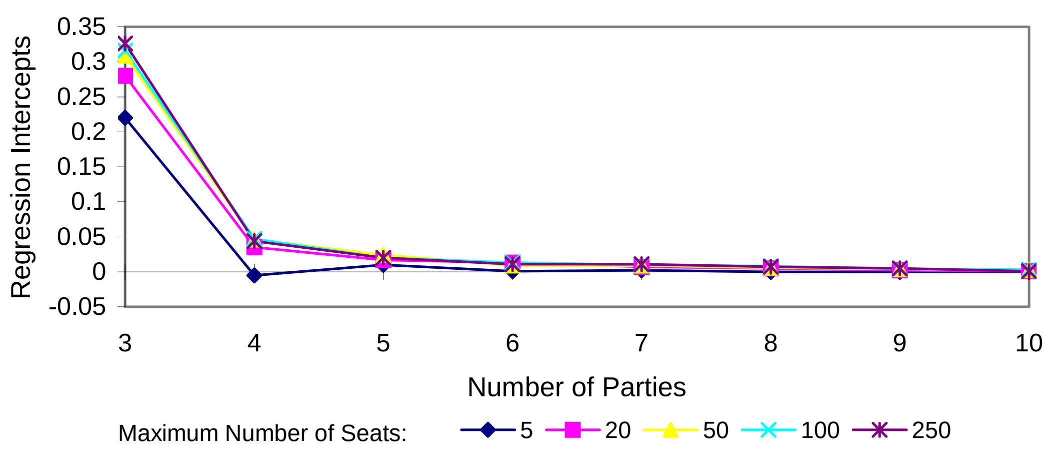


Figure 4: Weight Share Regression Intercepts



This effect is largest for smaller games since having few parties leads to only a handful of minimal winning coalitions. Thus, even if the smallest party is a member of only two minimal winning coalitions (as it is guaranteed to be for odd games)[8] then its share of coalition membership, and thereby power, will still be disproportionately large. Note that all three-player games reduce to either [2; 1, 1, 1] or [3; 2, 1, 1] (with the former being about eight times more common); thus, even the tiniest parties will have at least a 25% weight share.

[8] Naturally, this does not include dummy parties, which would not be members of any winning coalitions.

For games with six or more players, the power of the small party diminishes. As can be seen in Figures 3 and 4, both the coefficient and the constant values approach unity and zero as the games get larger. For many-seat games with 10 players—which avoid the "no reduction" problem of few-seat, 10-player games—there is almost perfect proportionality. Thus, for games with sufficiently many parties, minimum integer weights do represent a proportional mapping of *a priori* resources to power.

Could the bias toward solvable games influence this result? For three reasons, I do not believe so. First, a majority of games in each category was solved. Second, as alluded to earlier, the phenomenon of the minimum integer weights being the same as the original seats is non-existent in the larger games (only one game out of the 10,800 games with maximum seats of 50, 100, or 250 has this characteristic). Thus, all the games had to be reduced in some way. There is no clear connection between difficulty of reduction and disproportionality. Third, the trends toward unity and zero continue for the games with maximum seat size of 250, more of which were solved than the sets for seat size of 100 and 50. Even with somewhat incomplete data, the proportionality of minimum integer weights is verified.

4.4 Dummy Players

An interesting characteristic of games is the incidence of dummy players. While the minimum integer weights could not be found for all games, the coalitions were enumerated for each game; thus, there are data on dummy players for all 20,000 randomly generated games. After removing the dictator games, more than 17,000 games still remain in the analysis. In general, as shown in Figures 5 and 6, the percent of dummy players is negatively related to the number of parties and positively related to the maximum number of seats.

Figure 5: Incidence of Dummy Players by Party

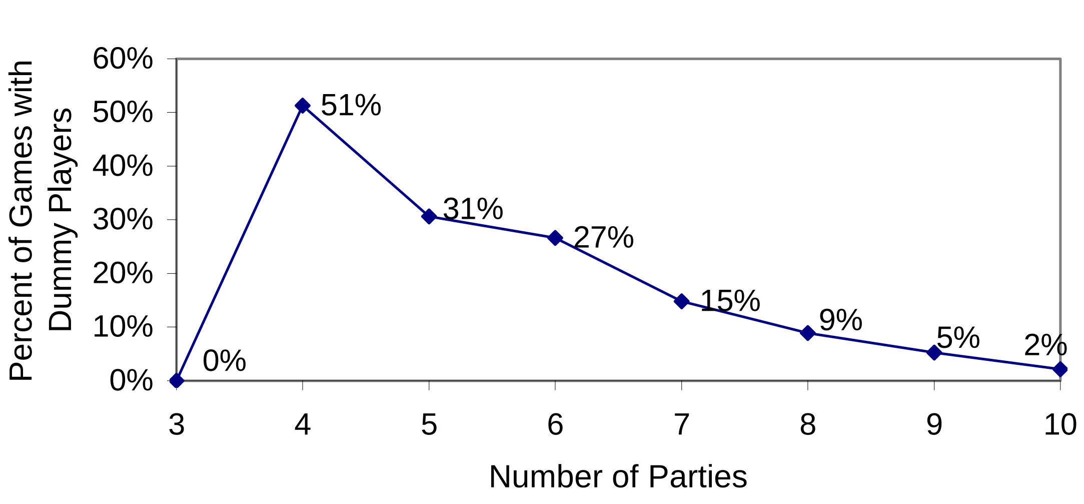
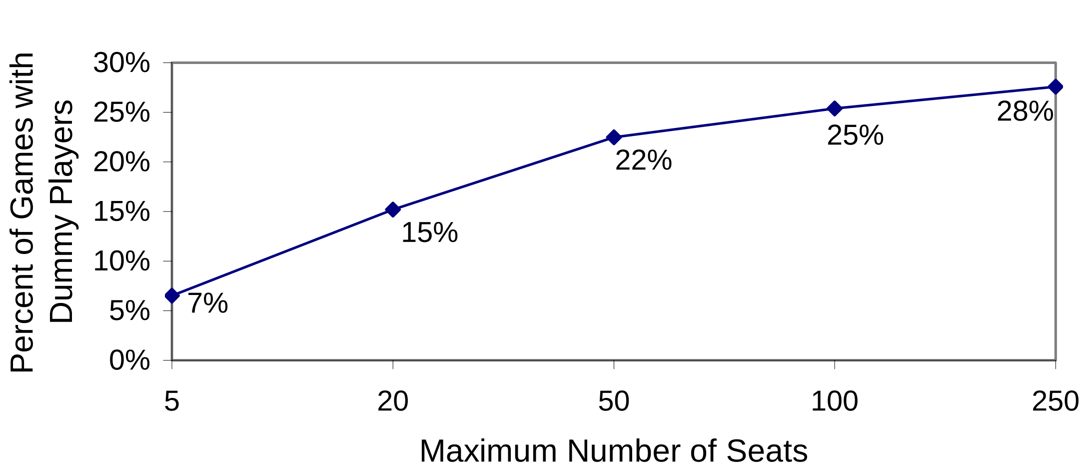


Figure 6: Incidence of Dummy Players by Maximum Seats



Other characteristics of games affect the frequency of dummy players. For instance, games with tying coalitions will *never* have a dummy player since each player will create a minimal winning coalition by adding itself to the side of the tying coalition it is not in. (Some other players in this new winning coalition might have to be dropped to create a minimal winning coalition.) Is coalition enumeration necessary to determine if there is a tying coalition? Probably, since deciding if there is a tying coalition is exactly the same as the well-known, NP-complete problem of PARTITION, which asks whether a set has two subsets of equal sums. Being NP-complete, no polynomial time algorithm is known to solve this problem.

An interesting question, therefore, is: What is the probability that if the total number of seats in the original game is even, there will be an even coalition? Of the 17,276 games examined, 8,034, or 47%, added up to an even number. Of these, 3,608, or 45%, had tying coalitions. However, this second percentage was significantly greater, 64%, for games of 8 players or more; also, the percentage monotonically increased as the number of players increased. Thus, one can guess that for large games, if the sum of the seats is an even number, there will be a tying coalition. As would be expected, all games with tying coalitions had minimal integer weights that added up to an even number; however, not all games that started with an even sum ended with "even-summed" minimum integer weights.

4.5 Problem Difficulty

Two potential measures of problem difficulty are: (1) the amount that the initial weights need to be reduced before they are minimal, and (2) the number of branch and bound nodes the algorithm must visit before finding integer weights. Clearly these will not be deterministically linked since some games, in which the algorithm gets "lucky" and finds the integer weights with the root node, might have much smaller minimum weights than *a priori* seats. A somewhat surprising result, however, is that the two variables have an extremely low correlation of 0.03. Figure 7 shows the two indicators in a scatterplot graph.

Certainly, given the current algorithm, the number of nodes visited is the best measure of how long the algorithm will take to minimize the game. An analysis determined which characteristics affect the number of nodes visited. First, the more parties and the more seats per party, the longer the algorithm will take to solve the puzzle.

Figure 7: Two Measures of Problem Difficulty

Second, for all games that are both homogeneous and odd,[9] the minimum integer weights are also the minimum weights; thus, the algorithm need only visit the root node. Third, even games usually take longer to solve, even when accounting for homogeneity.

	Types of Games	
	Heterogeneous	Homogeneous
Odd	1.001	1.000
Even	583.603	5.089

Table 8: Average Number of Nodes Visited

As Table 8 shows, these characteristics have large effects on game difficulty. Thus, one can estimate the number of nodes needed to visit before actually running the program. Before coalition enumeration, the following formula is the best estimator:

$$\text{Nodes-visited} = 24.88 * n - 0.07 * \text{max-seats} + 170.74 * \text{even-seats} - 160.74$$

[9] By odd, I mean games whose minimum integer weights sum to an odd number. This occurs if and only if there is no tying coalition.

Even-seats is a dummy variable, which takes the value of one when the seats in the original game add up to an even number. (The coefficient to max-seats is not statistically significant, but is included for completeness). The above equation explains only about 12% of the variance of the number of visited nodes.

The percent of variance explained jumps to 31% if, after coalition enumeration is completed, it is noted whether the game includes a tying coalition (i.e., whether sum of the minimum integer weights is even). The following best-fit equation can then be used to estimate nodes visited:

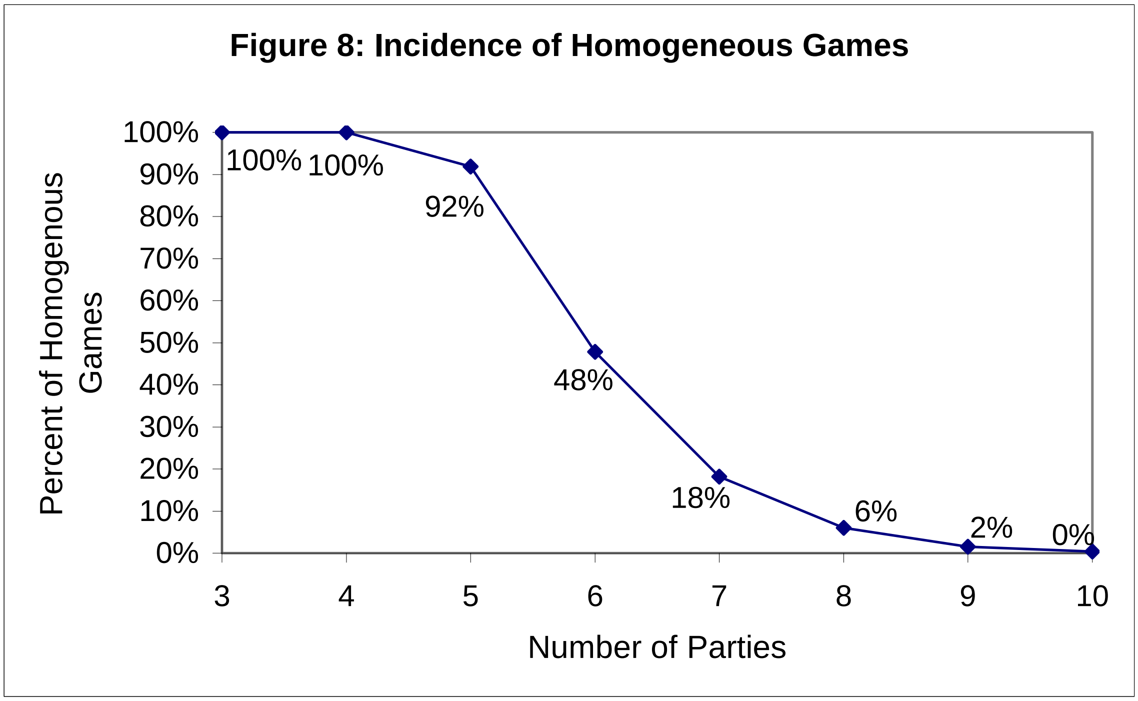
Nodes-visited = 12.41 * n + 0.45 * max-seats + 377.21 * even-weight – 122.72

Notice that the coefficient on max-seats becomes positive (and now statistically significant) when correcting for whether there is a tying coalition. This switch confirms the earlier hypothesis that many-seat games are easier to solve only because of an increased absence of tying coalitions. (Data on the mean number of nodes visited for each number of players and maximum seat combination are not given since the data would be severely biased in the downward direction, as harder games with many nodes to be visited were not solved.)

4.6 Homogeneity

The above analysis indicates that homogeneous games are easier to solve. Theorists have preferred dealing with homogeneous games ever since the 1950s, during which, for example, Isabel derived certain characteristics for homogeneous games only. Unfortunately, as Figure 8 demonstrates, the frequency of homogeneous games decreases precipitously as the number of parties increases.

At one extreme, all three- and four-player games are homogeneous. This trait quickly diminishes; for example, for games with at least seven players, the incidence of homogeneous games is under 20%. For games with ten or more players, finding a homogenous game is akin to the proverbial search for a needle in the haystack. Thus, for large games, scholars might assume homogeneity and produce a result with bounded error. That possibility begs the question of how similar heterogeneous games are to homogenous games.



To properly determine the potential benefits of such an estimator, a measure of heterogeneity is needed. For each minimal winning coalition, the difference between the coalition's weight and the quota is found. Then, to account for the differences in game size, this difference is divided by the total sum of the weights. This operation is performed on the minimum integer weights, not the original game. Let this value be called the surplus of the coalition. Formally,

Surplus for coalition S: $(\sum_{i \in S} w_i - q) / (\sum w_i)$

By the definition of homogeneity, homogenous games include only coalitions with surpluses of zero. Thus, the degree of a game's heterogeneity is measured as the mean of its coalitions' surpluses. While heterogeneous games become ubiquitous in larger games, Table 9 indicates that games do not increase in degree of heterogeneity as game size grows.

Number of Players	Maximum Number of Seats					Total
	5	20	50	100	250	
5	5.2%	4.2%	4.1%	3.7%	4.0%	4.8%
6	3.8%	3.9%	3.6%	3.6%	3.4%	3.7%
7	3.4%	4.1%	3.7%	3.8%	3.6%	3.7%
8	3.4%	4.2%	4.0%	3.9%	3.9%	3.9%
9	2.8%	4.1%	4.2%	4.0%	4.0%	3.7%
10	2.4%	3.8%	4.1%	4.1%	4.2%	3.6%
Total	3.2%	4.0%	3.9%	3.9%	3.9%	3.7%

Table 9: Heterogeneity of Non-homogeneous Games

While the surplus' mean remains relatively constant over the size of games, the distribution of the surplus does not. As Table 10 highlights, a larger proportion of coalitions in smaller games has high surplus values (here "high" is defined as ≥ 10%). Lest one conclude that smaller games have more heterogeneity, Table 11 indicates that the larger games have relatively more coalitions with some surplus ("some" being defined as ≥ 1%). In both cases, after the maximum number of seats has surpassed 5, the effect of increasing the maximum is faint (Table 11 shows a slight downward trend).

Number of Players	Maximum Number of Seats					Total
	5	20	50	100	250	
5	35%	34%	32%	33%	33%	35%
6	25%	22%	22%	23%	24%	23%
7	16%	17%	16%	17%	16%	16%
8	8%	15%	14%	14%	14%	13%
9	6%	12%	13%	12%	13%	11%
10	5%	9%	10%	11%	12%	9%
Total	12%	15%	15%	15%	15%	14%

Table 10: Percent of Coalitions with Surplus Greater than 10%
(Non-homogeneous Games Only)

Number of Players	Maximum Number of Seats 5	20	50	100	250	Total
5	35%	34%	32%	33%	33%	35%
6	29%	31%	27%	27%	25%	28%
7	33%	42%	37%	36%	33%	36%
8	38%	52%	50%	47%	46%	46%
9	35%	61%	61%	59%	59%	53%
10	33%	67%	72%	71%	71%	60%
Total	34%	51%	49%	48%	49%	45%

Table 11: Percent of Coalitions with Surplus Greater than 1%
(Non-homogeneous Games Only)

These results offer a mixed message to those seeking to approximate games by assuming homogeneity. On one hand, the dearth of homogeneous large-party games and the significant amount of heterogeneity (about 4%) in these games means that constructing an appropriate approximation will most likely be difficult. But, the value of 4% seems stable and does not grow with the addition of players or seats; thus, if an approximation algorithm is devised, it will probably apply to arbitrarily large games.

4.7 Empirical Results

While a theoretical understanding of minimum integer weights is crucial to form a framework of the principles of coalition formation, the real purpose of determining power is applications to real world problems, such as legislatures. To this end, data were collected from the parliaments of 16 European countries for the years 1945-1988. There were 374 different governing coalitions for those parliaments in that time frame. However, many of those governments were the result of reorganization rather than elections. Reorganization doesn't change the *a priori* seats controlled by each party; thus, to avoid a bias toward unstable countries, only elections results are considered. The election data provide 218 parliamentary configurations; these games are contrasted with the stochastic data in Table 12.

The algorithm solved most of the parliamentary games—even a majority of the very large games. The weakness of the analysis is not the lack of solved games, but the lack of games in general. The highest number of games at any one player-level is 33 (five-player games). But, even with the small number of games, some trends are clear; the data are detailed in Table 12.

Parliamentary games are not significantly more likely than the stochastic data to have dummy players. This result is comforting since it indicates that voters are not throwing their support behind parties that have no chance of affecting the coalition-formation process. Such "expressive voting" occurs, of course, and the percent of dummy players doesn't trend toward zero as rapidly in the stochastic data; but, compared to other discrepancies between the two types of data, the difference of dummy players is minor.

A more intriguing result is that, for large games, the prevalence of homogenous games in the parliamentary data is at least a magnitude more than in the stochastic data. No simple explanation for this discrepancy exists. The best candidate cause is that parliaments are by nature efficient in coalition formation. Having surplus members of coalitions is of no use, and only adds to the burden of governing. Thus, parties that routinely cause this surplus have an incentive to either lower their vote (and retain most of their power) or slightly increase their vote and probably gain a disproportionate amount of power.

Number of parties	EU* Number of Games	# of Games Solved	Pct. Solved	Stochastic* Number of Games	# of Games Solved	Pct. Solved
3	16	16	100%	2,500	2,500	100%
4	28	28	100%	2,500	2,500	100%
5	33	33	100%	2,500	2,500	100%
6	32	32	100%	2,500	2,500	100%
7	19	19	100%	2,500	2,496	100%
8	24	24	100%	2,500	2,345	94%
9	21	20	95%	2,500	1,913	77%
10	19	17	89%	2,500	1,837	73%
11+	26	17	65%	--	--	--
Total	218	206	94%	20,000	18,591	93%

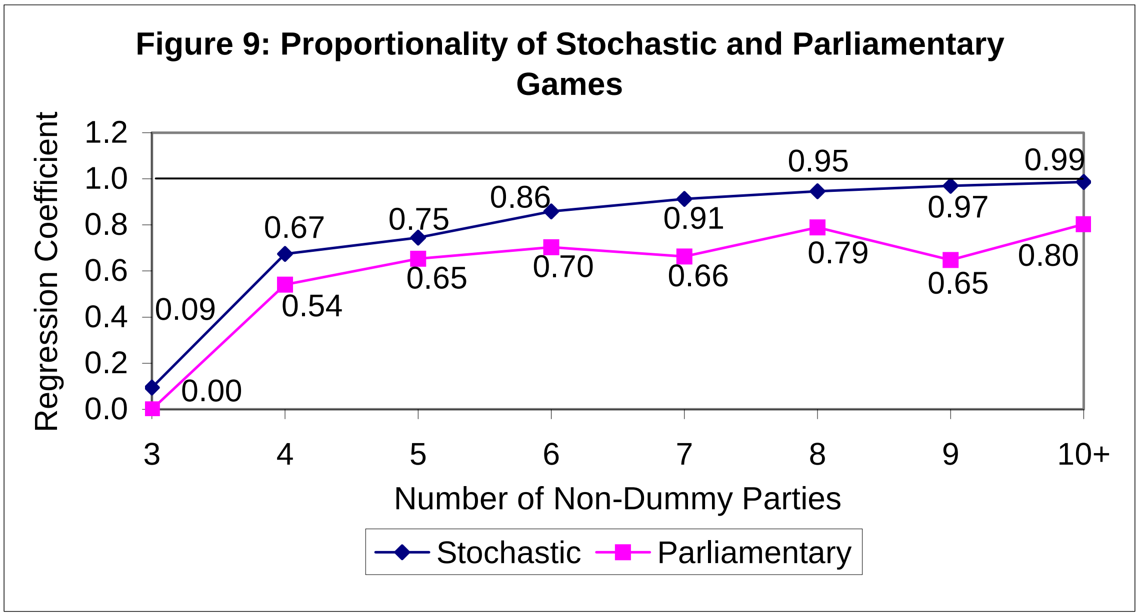
Number of parties	EU Pct. Have Dummy	Pct. Hom-ogenous	Mean Het-ergeneity	Stochastic Pct. Have Dummy	Pct. Hom-ogenous	Mean Het-ergeneity
3	0%	100%	--	0%	100%	--
4	33%	100%	--	51%	100%	--
5	41%	97%	4%	31%	92%	5%
6	30%	81%	2%	27%	48%	4%
7	19%	44%	3%	15%	18%	4%
8	11%	32%	3%	9%	6%	4%
9	6%	33%	2%	5%	2%	4%
10	6%	13%	2%	2%	0%	4%
11+	13%	19%	2%	--	--	--
Total	21%	61%	2%	19%	45%	4%

Number of parties	EU Regression Coefficient	Regression Constant	R-Squared	Stochastic Regression Coefficient	Regression Constant	R-Squared
3	0.000	0.333	1.000	0.142	0.286	0.142
4	0.533	0.117	0.500	0.857	0.036	0.621
5	0.720	0.056	0.777	0.904	0.019	0.760
6	0.734	0.044	0.805	0.939	0.010	0.858
7	0.699	0.043	0.839	0.940	0.009	0.929
8	0.832	0.021	0.915	0.954	0.006	0.965
9	0.668	0.037	0.787	0.972	0.003	0.984
10	0.805	0.019	0.933	0.987	0.001	0.994
11+	0.855	0.012	0.876	--	--	--
Total	0.741	0.037	0.797	0.913	0.013	0.854

All coefficients and intercepts > .001 are significant at the 99% confidence level.
*****The “games solved” analysis includes dictator games.

Table 12: Characteristics of Parliamentary Games vs. Stochastic Games

The proportionality result is not as clear for parliamentary games as it is for simulated data. One potential source of error is that while the overall percent of dummy players is approximately equal for both types of games, the percent seesaws more in the empirical data. To remove this noise, a similar analysis was done for non-dummy parties. Games with dummy parties were not excluded; they were just classified differently (*n* in this case being the number of non-dummy parties). The discrepancy still remains and is sizable, as illustrated in Figure 9.



A hint for why this difference occurs is in the Snyder, Ting, and Ansolabehere (2001) analysis of infinite games. Many political systems will have a few dominant parties and then several smaller regional, ethnic, or ideological parties. The quota is relatively low for the larger parties; thus, these games have "corner" equilibria. Smaller parties have disproportionately more weight in corner equilibria, which leads to the slower rise in the coefficient for parliamentary games. This conjecture is verified by the data, as the average ratio of the largest minimum integer weight to the quota is 60% for empirical games and 41% for stochastic games (for games with seven or more parties).

4.8 Comparison to Other Power Indices

Decades of research on voting games have yielded over a dozen different ways to measure power. Three such power indices, Shapley-Shubik, Banzhaf, and Deegan-Packel are contrasted with each other and with minimum integer weights. The main finding is that the Shapley-Shubik and Banzhaf indices are proportional while Deegan-Packel's index is wildly disproportionate.

An understanding of "swing" voters (or players) is necessary to calculate the Shapley-Shubik index. A swing voter is a player in a winning coalition (not necessarily minimal) whose weight is needed to keep the coalition winning. From this definition it follows that minimal winning coalitions are composed solely of swing voters. Shapley and Shubik also add a twist to this definition by ordering the parties in each coalition and repeating coalitions for each distinct ordering. For each coalition-ordering, if the last party is swing, then that party receives a Shapley-Shubik "point." At the end of this process, a party's Shapley-Shubik index value is its share of the total points.

The Banzhaf index is similar to the Shapley-Shubik index, and this is borne out in the statistical analysis. The main difference between these two indices is that Banzhaf's coalitions are not ordered. A party receives a Banzhaf "point" for each coalition in which it is a swing voter. As with the Shapley-Shubik index, a party's share of points is its end value.

The Deegan-Packel index is counter-intuitive because it can assign a higher index value to parties with fewer seats. This oddity stems from Deegan-Packel's use of minimal winning coalitions instead of winning coalitions that contain at least one swing voter. A party's Deegan-Packel "score" is the summation of the inverses of the size of each minimal winning coalition to which the party belongs. For instance, consider the game [7; 4, 3, 3, 1, 1]. The minimal winning coalitions (including duplicates) are: [4,3], [4,3], [3,3,1], and [3,3,1]. The Deegan-Packel index is calculated as follows:

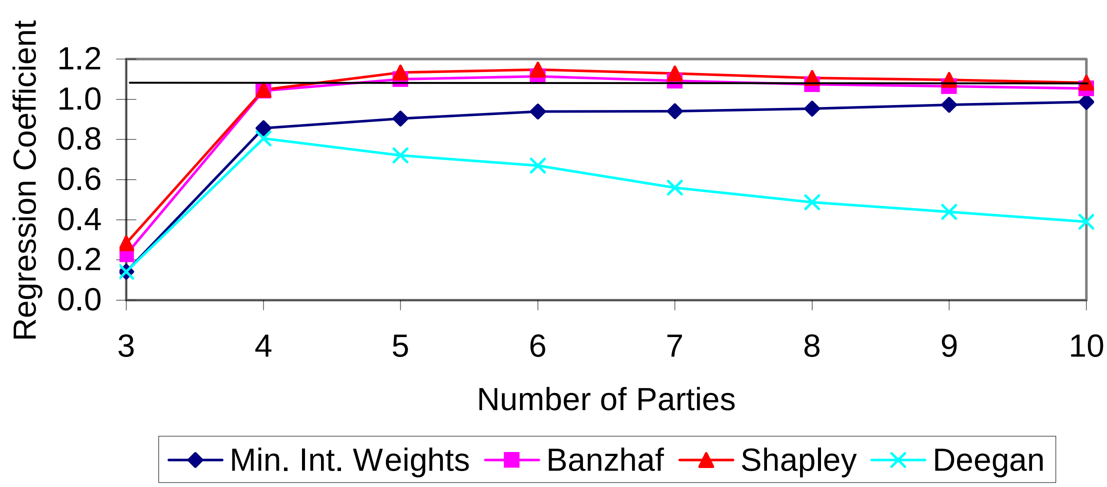
Party Seats	Score Summation	Total Score	Index
4	1/2 + 1/2	1.00	0.25
3	1/2 + 1/3 + 1/3	1.17	0.29
3	1/2 + 1/3 + 1/3	1.17	0.29
1	1/3	0.33	0.08
1	1/3	0.33	0.08
Total		4.00	1.00

Table 13: Deegan-Packel Calculation Example

In this case, the largest party does not have the largest index value. It is completely illogical that increasing a player's seat share should *decrease* that player's power. Yet the Deegan-Packel index implies this relationship in many cases.

A simple regression analysis confirms the intuition of disproportionate power for the Deegan-Packel index. Figures 10 and 11 show the trend in the regression coefficients and intercepts for the different power measures. The minimum integer weights, the Banzhaf index, and the Shapley-Shubik index act similarly as they all approach proportionality as the games get large. But, the Deegan-Packel index goes wildly astray,

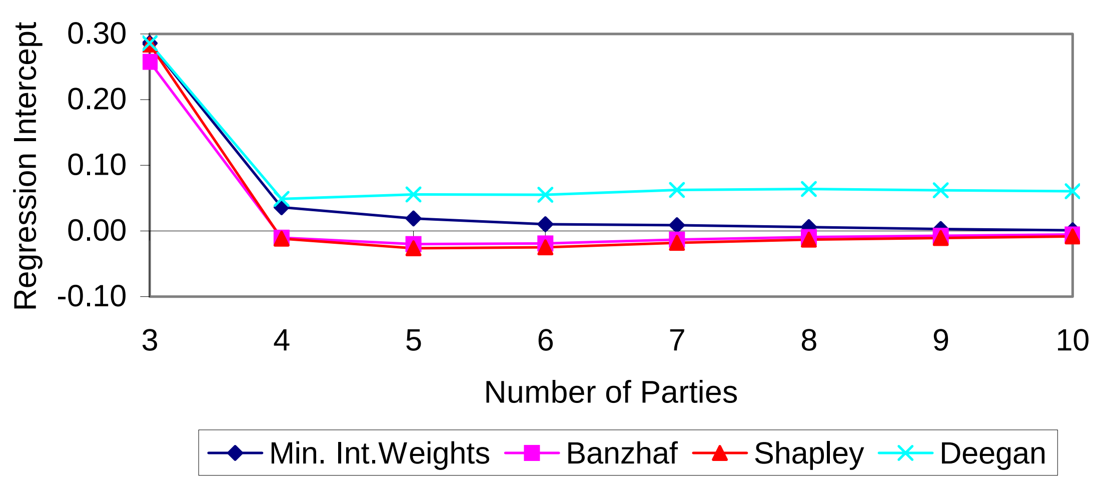
Figure 10: Power Indices' Regression Coefficients



introducing more disproportionality than even the *formateur* theories of David Baron would allow.

There are only minute differences between Banzhaf and Shapley-Shubik values, as might be expected since their computation is similar. For all games sizes, however, Shapley-Shubik is slightly more disproportionate than Banzhaf. For games of greater than five players, the Banzhaf and Shapley-Shubik indices ascribe disproportionate power to large parties, while the minimum integer weight value does the opposite. This distinction could lead to dilemmas of which power measure to use for medium-sized games.

Figure 11: Power Indices' Regression Intercepts



4.9 Case Study: The Nice Accord

In December 2000, the 15 members of the European Union, after a grueling 4-day summit, reached an agreement on the voting allocation for member states in the EU's Council of Ministers (Bryant 2000). Table 14 shows the voting scheme and the power indices' values for each country. As might be expected, the voting distribution is minimum integer; if it weren't, then the countries would probably just decide to use the minimal (and simpler) scheme.

For calculating the minimum integer weights, there are seven ranks (one for each distinct vote count). There are 77 unique-rank coalitions, with no tying coalitions. Table 14 shows that the Deegan-Packel index, while hardly differentiating between the power of Germany and Ireland, does monotonically decrease with vote share. The three other power measures differ by at most 0.3% for all countries.

Country	Population*	Population Share	Votes	Rank	Vote Share	Min. Int. Weights Share	Shapley-Shubik	Banzhaf	Deegan-Packel
Germany	83,251,851	21.9%	29	1	12.2%	12.2%	12.6%	12.5%	7.3%
UK	59,778,002	15.7%	29	1	12.2%	12.2%	12.6%	12.5%	7.3%
France	59,765,983	15.7%	29	1	12.2%	12.2%	12.6%	12.5%	7.3%
Italy	57,715,625	15.2%	29	1	12.2%	12.2%	12.6%	12.5%	7.3%
Spain	40,077,100	10.6%	27	2	11.4%	11.4%	11.7%	11.6%	7.1%
Netherlands	16,067,754	4.2%	13	3	5.5%	5.5%	5.5%	5.5%	6.8%
Greece	10,645,343	2.8%	12	4	5.1%	5.1%	4.9%	4.9%	6.6%
Belgium	10,274,595	2.7%	12	4	5.1%	5.1%	4.9%	4.9%	6.6%
Portugal	10,084,245	2.7%	12	4	5.1%	5.1%	4.9%	4.9%	6.6%
Sweden	8,876,744	2.3%	10	5	4.2%	4.2%	4.1%	4.2%	6.5%
Austria	8,169,929	2.2%	10	5	4.2%	4.2%	4.1%	4.2%	6.5%
Denmark	5,368,854	1.4%	7	6	3.0%	3.0%	2.7%	2.7%	6.4%
Finland	5,183,545	1.4%	7	6	3.0%	3.0%	2.7%	2.7%	6.4%
Ireland	3,883,159	1.0%	7	6	3.0%	3.0%	2.7%	2.7%	6.4%
Luxembourg	448,569	0.1%	4	7	1.7%	1.7%	1.5%	1.6%	4.6%
Total	379,591,298	100%	237		100%	100%	100%	100%	100%
Std. Deviation		0.07			0.04	0.04	0.04	0.04	0.01

*July 2002 estimate from *CIA Factbook*

Table 14: Voting Allocation of the Nice Accord

5 Future Work and Conclusion

While this analysis sheds insight into the characteristics and proper uses of minimum integer weights, some work remains to be done. Foremost, the stochastic analysis is incomplete since the current algorithm cannot process such a large quantity of games in a reasonable amount of time. Thus, ideas for efficiencies must be incorporated into new versions of the algorithm. Second, only power indices based on coalition enumeration were analyzed, not more complicated bargaining models. Third, some theoretical proofs on the properties of minimum integer weights are incomplete.

With respect to improving the algorithm, the constrained branch and bound algorithm detailed in Section 3.6 show great promise for solving the "search" half of the problem. As Matsui proved, coalition enumeration is NP-hard, suggesting that the "coalition enumeration" half of the problem will pose the greater challenge. Most likely, a paradigm shift will be required to find an efficient solution. Certainly one cannot simply enumerate all the coalitions (as is currently done), since those grow on the order of 2^n. A game will have to be represented by a polynomial number of coalitions; instead of picking those coalitions out of a pool of all coalitions, an expert or randomized algorithm will have to enumerate only the necessary coalitions. Bohossian and Bruck (1995) briefly touch upon this point and mention one possible solution in their early research.

Theoretically, Baron's bargaining model ascribes disproportionate power to the *formateur*. Implementing this model (and perhaps others that are similar, such as Morrelli's) is a tedious process of discovering and coding the underlying algorithm behind the theory. The results of such an analysis were not a top priority for the current research, and thus this procedure was postponed. Future work will detail how different game and seat sizes affect the disproportionality of *formateur* models.

Two proofs (or perhaps counterexamples) are lacking from the analysis of minimum integer weights. The first open question is whether interchangeable parties must have the same minimum integer weight. The beginnings of such a proof are noted in Section 3.2. If a counterexample is found, the current algorithm must use unique seats instead of ranks as the variables in the search for minimality. Also, while not as essential to this research, the question of whether minimum integer weights are strictly monotonic piques my curiosity and will be researched further.

While these questions are yet unfinished, the journey to this point has yielded many successes. An algorithm to calculate the minimum integer weights for an arbitrary game was developed. Though this algorithm scales exponentially with the size of the problem, given enough resources, any game can be solved using the methods outlined. Indeed, over 90 percent of real-world parliamentary games were solved quickly.

With this algorithm in hand, the properties of minimum integer weights were either discovered or confirmed. As the theoretical literature expects, the minimum integer weights are proportional to the *a priori* resources for large games. Also, homogeneous games become increasing rare in random large games, although they retain a strong presence in similarly sized parliaments. Dummy players are also increasingly uncommon in games with more players, but have a better chance of popping up if the number of total votes is large.

The Shapley-Shubik and Banzhaf indices are also proportional for large games, but approach proportionality from the opposite direction from the minimum integer weights. The Deegan-Packel index is flawed: it sometimes predicts that an increase in seats will lead to a decrease in power. The three measures of minimum integer weights, Shapley-Shubik, and Banzhaf all treated the European Union's voting scheme similarly.

With this algorithm, empirical researchers can calculate the minimum integer weights for their data points and use the results as the predicted proportional power share. No longer will scholars such as Warwick and Druckman be forced to use seat share as a stand in for proportional power, which can lead to unreasonable results. The formation of this algorithm is one step further in the quest to discover the relationship between resources and power.

Appendix A: Technical Optimizations for the Algorithm

Optimization 1: Using elementary row operations to keep the matrix B^{-1} updated in the execution of simplex. A straightforward approach would, after introducing a new variable into the basis matrix B, take the inverse of B to update B^{-1}. However, taking the inverse of a matrix is a very costly operation. A much more efficient solution is to keep track of the elementary row operation performed on B, and perform those operations on B^{-1}. In fact, the matrix B does not need to be stored; it is sufficient to hold, for each basis variable, which column of B that variable would be in.

Optimization 2: Rational numbers and GCD calculation. Most programming languages do not have exact floating point arithmetic. Thus, each real number needs to be stored as a rational number (i.e., a pair of two integers). To simplify a rational number, the greatest common denominator (GCD) of the numerator and denominator must be calculated. It is efficient to determine the GCD of a rational number only when (1) the number is needed or (2) the numerator and denominator are so large that an overflow will occur in the near future.

Optimization 3: Sorting the queue in branch and bound. A small optimization is to check whether a node is integral, and give it highest priority in the queue for its weight-level. Thus, one does not have to examine non-integral solutions of the same weight because the minimum integer weights would be found first.

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